

CONFORMAL FRAMES AND D-DIMENSIONAL GRAVITY

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We review some results concerning the properties of static, spherically symmetric solutions of multidimensional theories of gravity: various scalar-tensor theories and a generalized string-motivated model with multiple scalar fields and fields of antisymmetric forms associated with p -branes. A Kaluza-Klein type framework is used: there is no dependence on internal coordinates but multiple internal factor spaces are admitted. We discuss the causal structure and the existence of black holes, wormholes and particle-like configurations in the case of scalar vacuum with arbitrary potentials as well as some observational predictions for exactly solvable systems with p -branes: post-Newtonian coefficients, Coulomb law violation and black hole temperatures. Particular attention is paid to conformal frames in which the theory is initially formulated and which are used for its comparison with observations; it is stressed that, in general, these two kinds of frames do not coincide.

1. Introduction

The known gravitational phenomena are rather well described in the framework of conventional general relativity (GR). However, in a more general context of theoretical physics, whose basic aims are to construct a “theory of everything” and to explain why our Universe looks as it looks and not otherwise, most of the recent advances are connected with models in dimensions greater than four: Kaluza-Klein type theories, 10-dimensional superstring theories, M-theory and their further generalizations. Even if such theories (or some of them) successfully explain the whole wealth of particle and astrophysical phenomenology, there remains a fundamental question of finding direct observational evidence of extra dimensions, which is of utmost importance for the whole human world outlook.

Observational “windows” to extra dimensions are discussed for many years [1]–[3]. Among the well-known predictions are variations of the fundamental physical constants on the cosmological time scale [4]–[9]. Such constants are, e.g., the effective gravitational constant G and the fine structure constant α . There exist certain observational data on G stability on the level of $\Delta G/G \sim 10^{-11} \div 10^{-12} \text{ y}^{-1}$ [4, 5, 10], which restrict the range of viable cosmological models. Some evidence on the variability of α has also appeared from quasar absorption spectra: $\Delta\alpha/\alpha \sim -0.72 \cdot 10^{-5}$ over the redshift range $0.5 < z < 3.5$ [11] (the minus means that α was smaller in the past).

Other possible manifestations of extra dimensions include excitations in compactified factor spaces [12], which can behave as particles with a large variety of masses and contribute to dark matter or to cross-sections of usual particle interactions; monopole modes in gravitational waves; various predictions for standard cosmological tests and generation of the cosmological constant [13], and numerous effects connected with local field sources, including, in particular, deviations from the Newton and Coulomb laws [7, 8, 2, 14, 15] and the properties of black holes, especially in the actively discussed brane-world framework [16].

In this paper we discuss solutions of multidimensional theories of gravity of Kaluza-Klein type, i.e., under the condition that neither the metric nor other fields depend on the additional (internal) coordinates [1]–[3]. The 4D metric is generally specified in such theories up to multiplying by a conformal factor depending on scalar fields and extra-dimension scale factors. This is the well-known problem of choice of a physical conformal frame (CF). Mathematically, a transition from one CF to another is nothing else but a substitution in the field equations, which can be solved using any variables. However, physical predictions about the behaviour of matter (except massless particles) are CF-dependent.

Among possible CFs one is distinguished: the so-called Einstein frame, in which the metric field Lagrangian contains the scalar curvature \mathcal{R} with a constant coefficient. In other, so-called Jordan frames, \mathcal{R} appears with field-dependent factors.

The choice of a physical CF in non-Einsteinian theories of gravity is rather widely discussed, but mostly in four dimensions in the context of scalar-tensor theories (STT) and in higher-order theories with curvature-nonlinear gravitational Lagrangians — see, e.g., [17, 18, 19] and numerous references therein. The review [18] classified the authors of published papers by their attitude to the problem: those (i) neglecting the issue; (ii) supporting the view that all frames are equivalent; (iii) recognizing the problem but giving no conclusive arguments; (iv) claiming that a Jordan frame is physical; (v) asserting that the Einstein frame is physical. Each group included tens of names, and some names even got into more than one group.

Refs. [17] and [18] have presented arguments in favour of the Einstein frame, and the most important ones, applicable to STT and higher-order theories (and multiscalar-tensor theories obtainable from multidimensional gravity) are connected with the positivity of scalar field energy and the existence of a classically stable ground state.

In our view [5, 20, 21] [which turns out to be outside the groups (i)–(v)], the above arguments could be convincing if we dealt with an “absolute”, or “ultimate” theory of gravity. If, however, the gravitational action is obtained in a certain limit of a more fundamental unified theory, theoretical requirements like the existence of a stable ground state should be addressed to this underlying theory rather than its visible manifestation. In the latter, the notion of a physical CF should be only related to the properties of instruments used for measuring masses, lengths and time intervals. Moreover, different sets of instruments (different measurement systems [5]) are described, in general, by different CFs. We thus suppose that there can be at least two different physical CFs: the fundamental one, in which the underlying field theory (or a field limit of a more fundamental theory) is specified, and the observational one, corresponding to a given set of instruments. One can say that the first CF describes what is happening “as a matter of fact”, the second one — what we see.

The set of references used in the present observations is connected with atomic units, and the corresponding observational CF for any underlying theory is therefore the CF that provides geodesic motion for ordinary massive (fermionic) matter in 4 dimensions.

In what follows, we will first discuss the CF dependence of the properties of space-time using, as an example, static, spherically symmetric scalar field configurations in STT (Sec. 2) and in multidimensional theories of gravity with multiple factor spaces (Sec. 3). We shall see that some general theorems, valid in one CF, may be violated in another, and there are such conformal mappings that the space-times of different frames are even not in a one-to-one correspondence (the so-called conformal continuation [22]). Then, in Sec. 4, we will discuss the CF dependence of some observable quantities for a class of solutions of a generalized field model [23]–[26], containing multiple scalar fields and antisymmetric forms, associated with charged p -branes. This choice is motivated by the bosonic sector of the low-energy field approximation of superstring theories, M-theory and their generalizations [27]–[31]. The model is, however, not restricted to known theories

since it assumes arbitrary dimensions of factor spaces, arbitrary ranks of antisymmetric forms and an arbitrary number of scalar fields. Among the quantities to be discussed are (1) the post-Newtonian (PN) coefficients describing the weak field behaviour of the solutions, (2) for black hole solutions, the Hawking temperature T_{H} which is obviously important for small (e.g., primordial) black holes rather than those of the stellar or galactic mass range and (3) the parameters of Coulomb law violation for the 4D components of the antisymmetric forms which behave as an electromagnetic field. We do not fix the underlying fundamental theory and thus have no reason to prescribe a particular CF, therefore the results are formulated in an arbitrary frame.

2. Scalar-vacuum configurations in STT

2.1. STT in Jordan and Einstein frames

Consider a general (Bergmann-Wagoner-Nordtvedt) STT, in a D -dimensional manifold $\mathbb{M}_{\text{J}}[g]$ with the metric $g_{\mu\nu}$ (to be called the Jordan conformal frame), for which the gravitational field action is written of the form

$$S_{\text{STT}} = \int d^D x \sqrt{g} [f(\phi) \mathcal{R} + h(\phi) (\partial\phi)^2 - 2U(\phi)], \quad (1)$$

where $g = |\det(g_{\mu\nu})|$, $(d\phi)^2 = g^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi)$ and f , h , U are arbitrary functions of the scalar field ϕ .

The action (1) can be simplified by the well-known conformal mapping which generalizes Wagoner's [32] 4-dimensional transformation,

$$g_{\mu\nu} = F(\phi) \bar{g}_{\mu\nu}, \quad F(\phi) := |f(\phi)|^{-2/(D-2)}, \quad (2)$$

$$\frac{d\psi}{d\phi} = \pm \frac{\sqrt{|l(\phi)|}}{f(\phi)}, \quad l(\phi) := fh + \frac{D-1}{D-2} \left(\frac{df}{d\phi} \right)^2, \quad (3)$$

removing the nonminimal scalar-tensor coupling expressed in the factor $f(\phi)$ before \mathcal{R} . The action (1) is now specified in the new manifold $\mathbb{M}_{\text{E}}[\bar{g}]$ with the metric $\bar{g}_{\mu\nu}$ (the Einstein frame) and the new scalar field ψ :

$$S_{\text{E}} = \int d^D x \sqrt{\bar{g}} \left\{ \text{sign } f [\bar{\mathcal{R}} + (\text{sign } l) (\partial\psi)^2] - 2V(\psi) \right\}, \quad (4)$$

where the determinant \bar{g} , the scalar curvature $\bar{\mathcal{R}}$ and $(\partial\psi)^2$ are calculated using $\bar{g}_{\mu\nu}$, and

$$V(\psi) = |f(\phi)|^{-D/(D-2)} U(\phi). \quad (5)$$

The action (4) is similar to that of GR with a minimally coupled scalar field ψ but, in addition to arbitrary D , contains two sign factors. The usual sign of gravitational coupling corresponds to $f > 0$. On the other hand, theories with $l(\phi) < 0$ lead to an anomalous sign of the kinetic term of the ψ field in (4) — a “ghost” scalar field as it is sometimes called. Such fields violate all standard energy conditions and therefore easily lead to unusual solutions like wormholes [33, 34]. We will adhere to theories with $l > 0$. However, $f < 0$ in some regions of \mathbb{M}_{J} will appear due to continuations to be discussed further.

Among the three functions of ϕ entering into (1) only two are independent since there is a freedom of transformations $\phi = \phi(\phi_{\text{new}})$. We assume $h \geq 0$ and use this freedom, choosing in what follows $h(\phi) \equiv 1$.

2.2. No-go theorems for the Einstein frame

Let us discuss some general properties of static, spherically symmetric scalar-vacuum configurations. We begin with the Einstein frame $\mathbb{M}_E(\bar{g})$ with the action (4) and put $\text{sign } f = \text{sign } l = 1$, thus obtaining D -dimensional GR with a minimally coupled scalar field ψ .

Choosing the radial coordinate ρ corresponding to the gauge condition $\bar{g}_{tt}\bar{g}_{\rho\rho} = -1$, we can write an arbitrary static, spherically symmetric metric in the form

$$ds_E^2 = A(\rho) dt^2 - \frac{d\rho^2}{A(\rho)} - r^2(\rho) d\Omega_{d_0}^2 \quad (6)$$

where $d_0 = D - 2$ and $d\Omega_{d_0}^2$ is the linear element on the sphere \mathbb{S}^{d_0} of unit radius. This gauge is preferable for considering Killing horizons, described as zeros of the function $A(\rho)$. The reason is that near a horizon ρ varies (up to a positive constant factor) like manifestly well-behaved Kruskal-like coordinates used for an analytic continuation of the metric. Thus, using this coordinate, which may be called *quasiglobal* [35], one can “cross the horizons” preserving the formally static expression for the metric.

Three independent field equations due to (4) for the unknowns $A(\rho)$, $r(\rho)$ and $\psi(\rho)$ may be written as follows:

$$(A'r^{d_0})' = -(4/d_0)r^{d_0}V; \quad (7)$$

$$d_0 r''/r = -\psi'^2; \quad (8)$$

$$A(r^2)'' - r^2 A'' = (d_0 - 2)r'(A'r - 2Ar') + 2(d_0 - 1), \quad (9)$$

where the prime denotes $d/d\rho$. These are three combinations of the Einstein equations; the scalar field equation $(Ar^{d_0}\psi')' = r^{d_0}dV/d\psi$ can be obtained as their consequence.

Eqs. (7)–(9) cannot be exactly solved for a given arbitrary potential $V(\psi)$ but make it possible to prove some important theorems telling us what can and what cannot be expected from such a system:

- A. The no-hair theorem [36, 37] claiming that asymptotically flat black holes cannot have nontrivial external scalar fields with nonnegative $V(\psi)$. In other words, in case $V \geq 0$, the only asymptotically flat black hole solution is characterized outside the horizon by $V \equiv 0$, $\psi = \text{const}$ and the Schwarzschild (or Tangherlini in case $d_0 > 2$) metric, i.e., in (6) $r \equiv \rho$ and $A = A(r) = 1 - 2m/r^{d_0-1}$, $m = \text{const}$.
- B. The generalized Rosen theorem [38, 39] asserting that particle-like solutions (i.e., asymptotically flat solutions with a regular centre) do not exist in case $V \geq 0$.
- C. The nonexistence theorem for regular configurations without a centre (wormholes, horns, flux tubes with $\psi \neq \text{const}$) [35].
- D. The causal structure theorem [35], asserting that the list of possible types of global causal structures (described by Carter-Penrose diagrams) for configurations with any potentials $V(\psi)$ and any spatial asymptotics is the same as the one for $\psi = \text{const}$, namely: Minkowski (or AdS), Schwarzschild, de Sitter and Schwarzschild–de Sitter.

These results will be referred to as Statements A, B, C, D, respectively.

Some comments are in order. Statement A is proved [36, 37] (see also [39] for $D > 4$) by finding an integral relation whose two parts have different signs unless the scalar field is trivial. There also exist no-hair theorems for black holes with de Sitter and anti-de Sitter asymptotics [40].

Statement B is proved in its most general form [39] by comparing two expressions for the mass: one written as an integral of the energy density and another given by the Tolman formula.

In Statement C, a *wormhole* is, by definition, a configuration with two asymptotics at which $r(\rho) \rightarrow \infty$, hence $r(\rho)$ must have at least one regular minimum. A *flux tube* is characterized by $r = \text{const} > 0$, i.e., it is a static $(d_0 + 1)$ -dimensional cylinder. A *horn* is a configuration that tends to a flux tube at one of its asymptotics, i.e., $r(\rho) \rightarrow \text{const} > 0$ at one of the ends of the range of ρ . The statement is proved [35] using Eq. (8): e.g., it leads to $r'' \leq 0$, which is incompatible with a regular minimum of $r(\rho)$.

A proof of Statement D [35, 41] rests on Eq. (9) which implies that the function $A(\rho)/r^2$ cannot have a regular minimum, therefore $A(\rho)$ can have at most two simple zeros around a static (R) region with $A > 0$ or one double zero separating two nonstatic (T) regions ($A < 0$).

It should be stressed that the validity of Statements C and D is independent of any assumptions on the shape and even sign of the potential $V(\psi)$ and on the particular form of the spatial asymptotic.

In cases admitted by the above theorems, black hole and particlelike solutions can be obtained, as is confirmed by known explicit examples. Thus, there exist: (1) black holes possessing nontrivial scalar fields (scalar hair), with $V \geq 0$, but with non-flat and non-de Sitter asymptotics [42]; (2) black holes with scalar hair and flat asymptotics, but partly negative potentials [41]; (3) configurations with a regular centre, a flat asymptotic and positive mass, but also with partly negative potentials [41].

2.3. No-go theorems for generic scalar-tensor solutions

In this section we discuss the possible validity of Statements A–D for STT solutions in a Jordan frame.

One can notice that when a space-time manifold $\mathbb{M}_E[\bar{g}]$ (the Einstein frame) with the metric (6) is conformally mapped into another manifold $\mathbb{M}_J[g]$ (the Jordan frame) equipped with the same coordinates according to the law (2), then a horizon $\rho = h$ in \mathbb{M}_E passes into a horizon of the same order in \mathbb{M}_J , a centre ($r = 0$) and an asymptotic ($r \rightarrow \infty$) in \mathbb{M}_E pass into a centre and an asymptotic, respectively, in \mathbb{M}_J if the conformal factor $F = F(\rho)$ is regular (i.e., finite, at least C^2 -smooth and positive) at the corresponding values of ρ . A regular centre passes to a regular centre and a flat asymptotic to a flat asymptotic under evident additional requirements.

The validity of Statements A–D in the Jordan frame thus depends on the nature of the conformal mapping (2) that connects $\mathbb{M}_J[g]$ with $\mathbb{M}_E[\bar{g}]$. Thus, if F vanishes or blows up at an intermediate value of ρ , there is no one-to-one correspondence between \mathbb{M}_J and \mathbb{M}_E . In particular, if a singularity in \mathbb{M}_E is mapped to a regular sphere S_{trans} in \mathbb{M}_J , then \mathbb{M}_J should be continued beyond this sphere, and we obtain, by definition, a *conformal continuation* (CC) from \mathbb{M}_E into \mathbb{M}_J [43, 22].

Such continuations can only occur for special solutions: to be removed by a conformal factor, the singularity should be, in a sense, isotropic. Moreover, the factor F should behave precisely as is needed to remove it.

In more generic situations, for given \mathbb{M}_E , there is either a one-to-one correspondence between the two manifolds, or the factor F “spoils” the geometry and creates a singularity in \mathbb{M}_J , that is, in a sense, \mathbb{M}_J is “smaller” than \mathbb{M}_E . In these cases Statement D is obviously valid in \mathbb{M}_J . This is manifestly true for STT with $f(\phi) > 0$.

Statement C cannot be directly transferred to \mathbb{M}_J in any nontrivial case $F \neq \text{const}$. In particular, minima of $g_{\theta\theta}$ (wormhole throats) can appear. Though, wormholes as global entities are impossible in \mathbb{M}_J if the conformal factor F is finite in the whole range of ρ , including the boundary values. Indeed, assuming that there is such a wormhole, we shall immediately obtain two large r

asymptotics and a minimum of $r(\rho)$ between them even in \mathbb{M}_E , in contrast to Statement C valid there.

Statements A and B can also be extended to \mathbb{M}_J for generic STT solutions, but here we will not concentrate on the details and refer to the papers [44] (see also Sec. 3).

Conformal continuations, if any, can in principle lead to new, maybe more complex structures.

2.4. Conformal continuations

A CC from \mathbb{M}_E into \mathbb{M}_J can occur at such values of the scalar field ϕ that the conformal factor F in the mapping (2) is singular while the functions f , h and U in the action (1) are regular. This means that at $\phi = \phi_0$, corresponding to a possible transition surface $\mathbb{S}_{\text{trans}}$, the function $f(\phi)$ has a zero of a certain order n . Then, in the transformation (3) near $\phi = \phi_0$ in the leading order of magnitude

$$f(\phi) \sim \Delta\phi^n, \quad n = 1, 2, \dots, \quad \Delta\phi \equiv \phi - \phi_0. \quad (10)$$

One can notice, however, that $n > 1$ leads to $l(\phi_0) = 0$ (recall that by our convention $h(\phi) \equiv 1$). This generically leads to a curvature singularity in \mathbb{M}_J , as can be seen from the trace of the metric field equation due to (1) [22]. We therefore assume $l > 0$ at $\mathbb{S}_{\text{trans}}$. Therefore, according to (3), we have near $\mathbb{S}_{\text{trans}}$ ($\phi = \phi_0$):

$$f(\phi) \sim \Delta\phi \sim e^{-\psi\sqrt{d_0/(d_0+1)}}, \quad (11)$$

where without loss of generality we choose the sign of ψ so that $\psi \rightarrow \infty$ as $\Delta\phi \rightarrow 0$.

In the CC case, the metric $\bar{g}_{\mu\nu}$ is singular on $\mathbb{S}_{\text{trans}}$ while $g_{\mu\nu} = F(\phi)\bar{g}_{\mu\nu}$ is regular. There are two opportunities. The first one, to be called CC-I for short, is that $\mathbb{S}_{\text{trans}}$ is an *ordinary regular surface* in \mathbb{M}_J , where both $g_{tt} = \mathcal{A} = FA$ and $-g_{\theta\theta} = R^2 = Fr^2$ (squared radius of $\mathbb{S}_{\text{trans}}$) are finite. (Here θ is one of the angles that parametrize the sphere \mathbb{S}^{d_0} .) The second variant, to be called CC-II, is that $\mathbb{S}_{\text{trans}}$ is a *horizon* in \mathbb{M}_J . In the latter case only $g_{\theta\theta}$ is finite, while $g_{tt} = 0$.

Given a metric $\bar{g}_{\mu\nu}$ of the form (6) in \mathbb{M}_E , a CC-I can occur if

$$F(\psi) = |f|^{-2/d_0} \sim 1/r^2 \sim 1/A \quad (12)$$

as $\psi \rightarrow \infty$, while the behaviour of f is specified by (11). In \mathbb{M}_E , the surface $\mathbb{S}_{\text{trans}}$ ($r^2 \sim A \rightarrow 0$) is either a singular centre, if the continuation occurs in an R-region, or a cosmological singularity in the case of a T-region.

It has been shown [22] that necessary and sufficient conditions for the existence of CC-I are that $f(\phi)$ has a simple zero at some $\phi = \phi_0$, and $|U(\phi_0)| < \infty$. Then there is a solution in \mathbb{M}_J , smooth in a neighbourhood of the surface $\mathbb{S}_{\text{trans}}$ ($\phi = \phi_0$), and in this solution the ranges of ϕ are different on different sides of $\mathbb{S}_{\text{trans}}$.

Thus any STT with $h \equiv 1$ admitting a simple zero of $f(\phi)$ admits a CC-I. The smooth solution in \mathbb{M}_J corresponds to two solutions on different sides of $\mathbb{S}_{\text{trans}}$ in two different Einstein frames. These solutions are special, being restricted by Eq. (12).

It is of interest that, under the CC-I conditions, any finite potential $V(\psi)$ is inessential near $\mathbb{S}_{\text{trans}}$: the solution is close to Fisher's scalar-vacuum solution [45] for $D = 4$ or its modification in other dimensions. For $U(\phi)$, the CC-I conditions do not lead to other restrictions than regularity at $\phi = \phi_0$.

In case $D = 3$, as follows from Eq. (9), a necessary condition for CC-I is $A/r^2 = \text{const}$.

A CC-II requires more special conditions [22], namely, there should be $D \geq 4$, $U(\phi) = 0$ and $dU/d\phi \neq 0$ at $\phi = \phi_0$. It then follows that $\mathbb{S}_{\text{trans}}$ is a second-order horizon, connecting two T-regions

in \mathbb{M}_J . Thus the only kind of STT configurations admitting CC-II is a $D \geq 4$ Kantowski-Sachs cosmology consisting of two T-regions (in fact, epochs, since ρ is then a temporal coordinate), separated by a double horizon.

2.5. Global properties of continued solutions

A solution to the STT equations may *a priori* undergo a number of CCs, so that each region of \mathbb{M}_J between adjacent surfaces $\mathbb{S}_{\text{trans}}$ is conformally equivalent to some \mathbb{M}_E . However, the global properties of \mathbb{M}_J with CCs are not so diverse as one might expect. In particular, Statement D, restricting possible causal structures, holds in \mathbb{M}_J in the same form as in \mathbb{M}_E .

A key point for proving this is the observation that the quantity $B = A/r^2$ is insensitive to conformal mappings (both its numerator and denominator are multiplied by F) and is thus common to \mathbb{M}_J and \mathbb{M}_E which is equivalent to a given part of \mathbb{M}_J . Therefore zeros and extrema of B inside \mathbb{M}_E preserve their meaning in the corresponding part of \mathbb{M}_J . Statement D rests on the fact that $B(\rho)$ cannot have a regular minimum in \mathbb{M}_E ; the same is true in a region of \mathbb{M}_J equivalent to some \mathbb{M}_E , and a minimum can only take place on a transition surface $\mathbb{S}_{\text{trans}}$ between such regions. A direct inspection shows [22] that this is not the case. Therefore Statement D is valid in \mathbb{M}_J despite any number of CC's.

As for Statements A–C, the situation is more involved. To our knowledge, full analogues of Statements A and B (probably with additional restrictions) for a sufficiently general STT are yet to be obtained (see, however, [44]). Statement C is evidently violated due to CCs since wormholes are a generic product of such continuations [22].

Indeed, a generic behaviour of \mathbb{M}_E is that r varies from zero to infinity. Let there be a family of such static solutions and let $f(\phi)$ have a simple zero. Then there is a subfamily of solutions admitting CC-I. A particular solution from this subfamily can come across a singularity beyond $\mathbb{S}_{\text{trans}}$ [due to $f(\phi) \rightarrow \infty$ or $l(\phi) \rightarrow 0$], but if “everything is quiet”, it will, in general, arrive at another spatial asymptotic and will then describe a wormhole.

It can be shown [22] that, under our assumption $l > 0$, there cannot be more than two values of ϕ where CCs are possible, i.e., where $f = 0$ and $df/d\phi \neq 0$. This does not mean, however, that an STT solution cannot contain more than two CCs. The point is that ϕ as a function of the radial coordinate is not necessarily monotonic, so there can be two or more CCs corresponding to the same value of ϕ . A transition surface $\mathbb{S}_{\text{trans}} \in \mathbb{M}_J$ corresponds to $r = 0$ in \mathbb{M}_E , therefore an Einstein-frame manifold \mathbb{M}_E , describing a region between two transitions, should contain two centres, more precisely, two values of the radial coordinate (say, ρ) at which $r = 0$. This property, resembling that of a closed cosmological model, is quite generic due to $r'' \leq 0$ in Eq. (8), but a special feature is that the conditions (12) should hold at both centres.

Well-known particular examples of CC-I are connected with massless nonminimally coupled scalar fields in GR, which may be described as STT with $f(\phi) = 1 - \xi\phi^2$, $h(\phi) = 1$, $U(\phi) = 0$. One such example is a black hole with a conformally coupled field ($\xi = 1/6$) [47, 36], such that $\phi = \infty$ but the energy-momentum tensor is finite on the horizon. Other examples are wormholes supported by conformal [33] and nonconformal [49] fields. Ref. [22] contains an example of a configuration with an infinite number of CCs, built using a conformally coupled scalar field with a nonzero potential in three dimensions.

3. Theories with multiple factor spaces

3.1. Reduction

In Sec. 2 we have been concerned with STT solutions in D -dimensional space-times with the metrics $\bar{g}_{\mu\nu}$ given by (6) and $g_{\mu\nu} = F(\phi)\bar{g}_{\mu\nu}$. Let us now pass to space-times \mathbb{M}^D with a more general structure

$$\mathbb{M}^D = \mathbb{R}_u \times \mathbb{M}_0 \times \mathbb{M}_1 \times \mathbb{M}_2 \times \cdots \times \mathbb{M}_n \quad (13)$$

where $\mathbb{M}_{\text{ext}} = \mathbb{R}_u \times \mathbb{M}_0 \times \mathbb{M}_1$ is the “external” manifold, $\mathbb{R}_u \subseteq \mathbb{R}$ is the range of the radial coordinate u , \mathbb{M}_1 is the time axis, $\mathbb{M}_0 = \mathbb{S}^{d_0}$. Furthermore, $\mathbb{M}_2, \dots, \mathbb{M}_n$ are “internal” factor spaces of arbitrary dimensions d_i , $i = 2, \dots, n$, and, according to this notation, we also have $\dim \mathbb{M}_0 = d_0$ and $\dim \mathbb{M}_1 = d_1 = 1$. The metric is taken in the form

$$ds_D^2 = -e^{2\alpha^0} du^2 - e^{2\beta^0} d\Omega_{d_0}^2 + e^{2\beta^1} dt^2 - \sum_{i=2}^n e^{2\beta^i} ds_i^2, \quad (14)$$

where ds_i^2 ($i = 2, \dots, n$) are metrics of Einstein spaces of arbitrary dimensions d_i and signatures while α^0 and all β^i are functions of the radial coordinate u .

Consider in \mathbb{M}^D a field theory with the action

$$S = \int d^D x \sqrt{|g_D|} \left[\mathcal{R}_D h_{ab}(\bar{\phi}) g^{MN} (\partial_M \phi^a) (\partial_N \phi^b) - 2V_D(\bar{\phi}) \right], \quad (15)$$

where \mathcal{R}_D is the D -dimensional scalar curvature and the scalar field Lagrangian has a σ -model form. We assume that ϕ^a are functions of the external space coordinates x^μ ($\mu = 0, 1, \dots, d_0 + 1$), so that actually in (15) $g^{MN} (\partial_M \phi^a) (\partial_N \phi^b) = g^{\mu\nu} (\partial_\mu \phi^a) (\partial_\nu \phi^b) \equiv (\partial \phi^a, \partial \phi^b)$, where the metric $g_{\mu\nu}$ is formed by the first three terms in (14). The metric h_{ab} of the N' -dimensional target space \mathbb{T}_ϕ , parametrized by ϕ^a , and the potential V are functions of $\bar{\phi} = \{\phi^a\} \in \mathbb{T}_\phi$.

The action (15) represents in a general form the scalar-vacuum sector of diverse supergravities and low-energy limits of string and p -brane theories [27, 28]. In many papers devoted to exact solutions of such low-energy theories (see Sec. 4) all internal factor spaces are assumed to be Ricci-flat, and nonzero potentials $V_D(\bar{\phi})$ are not introduced due to technical difficulties of solving the equations. Meanwhile, the inclusion of a potential not only makes it possible to treat massive and/or nonlinear and interacting scalar fields, but is also necessary for describing, e.g., the symmetry breaking and Casimir effects. (On the use of effective potentials for describing the Casimir effect in compact extra dimensions, see, e.g., [12] and references therein.)

Let us perform a dimensional reduction to the external space-time \mathbb{M}_{ext} with the metric $g_{\mu\nu}$. Eq. (15) is converted to

$$S = \int d^{d_0+2} x \sqrt{|g_{d_0+2}|} e^{\sigma_2} \left\{ \mathcal{R}_{d_0+2} + \sum_{i=2}^n d_i (d_i - 1) K_i e^{-2\beta^i} \right. \\ \left. + 2\nabla^\mu \nabla_\mu \sigma_2 + \sum_{i,k=2}^n (d_i d_k + d_i \delta_{ik}) (\partial \beta^i, \partial \beta^k) + L_{\text{sc}} \right\}, \quad (16)$$

where all quantities, including the scalar \mathcal{R}_{d_0+2} , are calculated with the aid of $g_{\mu\nu}$, and $\sigma_2 := \sum_{d=2}^n d_i \beta^i$, so that e^{σ_2} is the volume factor of extra dimensions.

It is helpful to pass in the action (1), just as in the STT (1), from the Jordan-frame metric $g_{\mu\nu}$ in \mathbb{M}_{ext} to the Einstein-frame metric

$$\bar{g}_{\mu\nu} = e^{2\sigma_2/d_0} g_{\mu\nu}. \quad (17)$$

Then, omitting a total divergence, one obtains the action (15) in terms of $\bar{g}_{\mu\nu}$:

$$S = \int d^{d_0+2}x \sqrt{|\bar{g}|} \left[\bar{\mathcal{R}} + H_{KL}(\partial\varphi^K, \partial\varphi^L) - 2V(\vec{\varphi}) \right]. \quad (18)$$

Here the set of fields $\{\varphi^K\} = \{\beta^i, \phi^a\}$, combining the scalar fields from (15) and the moduli fields β^i , is treated as a vector in the extended $N = (n-1+N')$ -dimensional target space \mathbb{T}_φ with the metric

$$(H_{KL}) = \begin{pmatrix} d_i d_k / d_0 + d_i \delta_{ik} & 0 \\ 0 & h_{ab} \end{pmatrix}, \quad (19)$$

while the potential $V(\varphi)$ is expressed in terms of $V_D(\vec{\phi})$ and β^i :

$$V(\vec{\varphi}) = e^{-2\sigma_2/d_0} \left[V_D(\vec{\phi}) - \frac{1}{2} \sum_{i=2}^n K_i d_i (d_i - 1) e^{-2\beta^i} \right]. \quad (20)$$

3.2. Extended no-go theorems

The action (18) brings the theory (15) to a form quite similar to (4) ($f > 0$), but a single field ψ is now replaced by a σ model with the target space metric H_{KL} . It can be easily shown [39] that Statements A–D are entirely extended to the theory (18) under the condition that the metric H_{KL} is positive-definite, which is always the case as long as h_{ab} is positive-definite.

Let us now discuss the properties of the D -dimensional metric g_{MN} given by (14). Its “external” part $g_{\mu\nu}$ is connected with $\bar{g}_{\mu\nu}$ by the conformal transformation (17). Since the action (15) corresponds to GR in D dimensions, this frame may be called the D -dimensional Einstein frame, and we will now designate the manifold \mathbb{M}^D endowed with the metric g_{MN} as \mathbb{M}_E^D .

The nonminimal coupling coefficient in the action (1), being connected with the extra-dimension volume factor e^{σ_2} , is nonnegative by definition, and the solution terminates where e^{σ_2} vanishes or blows up. Thus, in contrast to the situation in STT, conformal continuations are here impossible: one cannot cross a surface, if any, where e^{σ_2} vanishes. Roughly speaking, the Jordan-frame manifold $\mathbb{M}_{\text{ext}}[g]$ can be smaller but cannot be larger than $\mathbb{M}_{\text{ext}}[\bar{g}]$. If $\sigma_2 \rightarrow \pm\infty$ at an intermediate value of the radial coordinate, then the transformation (17) maps $\mathbb{M}_{\text{ext}}[g]$ to only a part of $\mathbb{M}_{\text{ext}}[\bar{g}]$.

Asymptotic flatness of the metric g_{MN} in \mathbb{M}_E^D implies an asymptotically flat Einstein-frame metric $\bar{g}_{\mu\nu}$ in \mathbb{M}_{ext} and finite limits of the moduli fields β^i , $i \geq 2$, at large radii. A similar picture is observed with the regular centre conditions: a regular centre in \mathbb{M}_E^D is only possible if there is a regular centre in $\mathbb{M}_{\text{ext}}[\bar{g}]$ and β^i , $i \geq 2$ sufficiently rapidly tend to constant values. A horizon in \mathbb{M}_E^D always corresponds to a horizon in $\mathbb{M}_{\text{ext}}[\bar{g}]$. (The opposite assertions are not always true, e.g., a regular centre in $\mathbb{M}_{\text{ext}}[\bar{g}]$ may be “spoiled” when passing to g_{MN} by an improper behaviour of the moduli fields β^i .)

So the global properties of $\mathbb{M}_{\text{ext}}[\bar{g}]$ and $\mathbb{M}_{\text{ext}}[g]$ (and hence \mathbb{M}_E^D), associated with Statements A–D, are closely related but not entirely coincide.

A. The **no-hair theorem** can be formulated for \mathbb{M}_E^D as follows:

Given the action (15) with h_{ab} positive-definite and a nonnegative potential (20) in the space-time \mathbb{M}_E^D with the metric (14), the only static, asymptotically flat black hole solution to the field equations is characterized in the region of outer communication by $\phi^a = \text{const}$, $\beta^i = \text{const}$ ($i = \overline{2, n}$), $V(\vec{\varphi}) \equiv 0$ and the Tangherlini metric $g_{\mu\nu}$.

In other words, the only asymptotically flat black hole solution is given by the Tangherlini metric in \mathbb{M}_{ext} , constant scalar fields ϕ^a and constant moduli fields β^i outside the event horizon. Note that

in this solution the metrics $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ in \mathbb{M}_{ext} are connected by simple scaling with a constant conformal factor since $\sigma_2 = \text{const}$.

Another feature of interest is that it is the potential (20) that vanishes in the black hole solution rather than the original potential $V_D(\bar{\phi})$ from Eq. (15). Theorem 5 generalizes the previously known property of black holes with the metric (14) when the internal spaces are Ricci-flat and the source is a massless, minimally coupled scalar field without a potential [46].

B. Particle-like solutions: Statement B is valid in \mathbb{M}_E^D in the same formulation as previously in \mathbb{M}_E , but the condition $V \geq 0$ now also applies to the potential (20) rather than $V_D(\bar{\phi})$ from (15).

C. Wormholes and even wormhole throats are impossible with the metric $\bar{g}_{\mu\nu}$. The conformal factor $e^{2\sigma_2/d_0}$ in (17) removes the prohibition of throats since for $g_{\mu\nu}$ a condition like $r'' \leq 0$ is no longer valid. However, a wormhole as a global entity with two large r asymptotics cannot appear in $\mathbb{M}_J = \mathbb{M}_{\text{ext}}[g_{\mu\nu}]$ for the same reason as in Sec. 2.3.

Flux-tube solutions with nontrivial scalar and/or moduli fields are absent, as before, but horns are not ruled out since the behaviour of the metric coefficient $g_{\theta\theta}$ is modified by conformal transformations.

It should be emphasized that all the restrictions mentioned in items A-C are invalid if the target space metric h_{ab} is not positive-definite.

D. The global causal structure of any Jordan frame cannot be more complex than that of the Einstein frame even in STT, where conformal continuations are allowed — see Sec. 2.5. The corresponding reasoning of [22] entirely applies to $\mathbb{M}_{\text{ext}}[g]$ and hence to \mathbb{M}_E^D . The list of possible global structures is again the same as that for the Tangherlini-de Sitter metric. This restriction does not depend (i) on the choice and even sign of scalar field potentials, (ii) on the nature of asymptotic conditions and (iii) on the algebraic properties of the target space metric. It is therefore the most universal property of spherically symmetric configurations with scalar fields in various theories of gravity.

A theory in \mathbb{M}^D may, however, be initially formulated in another conformal frame than in (15), i.e., with a nonminimal coupling factor $f(\bar{\phi})$ before \mathcal{R}_D . Let us designate M^D in this case as \mathbb{M}_J^D , a D -dimensional Jordan-frame manifold. (An example of such a construction is the so-called string metric in string theories [27, 28] where f depends on a dilaton field related to string coupling.) Applying a transformation like (2), we can recover the Einstein-frame action (15) in \mathbb{M}_E^D , then by dimensional reduction pass to $\mathbb{M}_{\text{ext}}[g]$ and after one more conformal mapping (17) arrive at the $(d_0 + 2)$ Einstein frame $\mathbb{M}_{\text{ext}}[\bar{g}]$. Addition of the first step in this sequence of reductions weakens our conclusions to a certain extent. The main point is that we cannot *a priori* require $f(\bar{\phi}) > 0$ in the whole range of $\bar{\phi}$, therefore conformal continuations (CCs) through surfaces where $f = 0$ are not excluded.

Meanwhile, the properties of CCs have only been studied [22] for a single scalar field in \mathbb{M}_{ext} (in the present notation). In our more complex case of multiple scalar fields and factor spaces, such a continuation through the surface $f(\bar{\phi}) = 0$ in the multidimensional target space \mathbb{T}_ϕ can have yet unknown properties.

One can only say for sure that the no-hair and no-wormhole theorems fail if CCs are admitted. This follows from the simplest example of CCs in the solutions with a conformal scalar field in GR, leading to black holes [47, 48] and wormholes [33, 49] and known since the 70s although the term “conformal continuation” was introduced only recently [43]. A wormhole was shown to be one of the generic structures appearing as a result of CCs in STT ([22], see sect 2.4 of the present paper).

If we require that the function $f(\bar{\phi})$ should be finite and nonzero in the whole range \mathbb{R}_u of the radial coordinate, including its ends, then all the above no-go theorems are equally valid in \mathbb{M}_E^D .

and \mathbb{M}_J^D . One should only bear in mind that the transformation (2) from \mathbb{M}_E^D to \mathbb{M}_J^D modifies the potential $V_D(\bar{\phi})$ multiplying it by $f^{-D/(D-2)}$, which in turn affects the explicit form of the condition $V \geq 0$, essential for Statements A and B.

Statement D on possible horizon dispositions and global causal structures will be unaffected if we even admit an infinite growth or vanishing of $f(\bar{\phi})$ at the extremes of the range \mathbb{R}_u . However, Statement C will not survive: such a behaviour of f may create a wormhole or horn in \mathbb{M}_J^D . A simple example of this kind is a “horned particle” in the string metric in dilaton gravity of string origin, studied by Banks et al. [50].

4. p -branes and observable effects

4.1. The model and the target space \mathbb{V}

Let us now consider a model which can be associated with p -branes as sources of antisymmetric form fields. Namely, in the space-time (13) with the metric (14), we take, as in Refs. [23]–[26], [51, 52], the model action for D -dimensional gravity with several scalar dilatonic fields φ^a and antisymmetric n_s -forms F_s :

$$S = \frac{1}{2\kappa^2} \int d^D z \sqrt{|g|} \left\{ \mathcal{R} + \delta_{ab} g^{MN} \partial_M \varphi^a \partial_N \varphi^b - \sum_{s \in \mathcal{S}} \frac{1}{n_s!} e^{2\lambda_{sa} \varphi^a} F_s^2 \right\}, \quad (21)$$

where $F_s^2 = F_{s, M_1 \dots M_{n_s}} F_s^{M_1 \dots M_{n_s}}$; λ_{sa} are coupling constants; $s \in \mathcal{S}$, $a \in \mathcal{A}$, where \mathcal{S} and \mathcal{A} are some finite sets. Essential differences from (15) are, besides the inclusion of the term with F_s^2 , that (i) the potential U is omitted (i.e., φ^a are not self-coupled but coupled to F_s), (ii) the “extra” spaces \mathbb{M}_i are assumed to be Ricci-flat and (for simplicity) spacelike and (iii) the target space metric h_{ab} is Euclidean, $h_{ab} = \delta_{ab}$. The “scale factors” e^{β^i} and the scalars φ^a are again assumed to depend on u only.

The F -forms (or, more precisely, their particular nonzero components, fixed up to permutation of indices, to be labelled with the subscript s) should also be compatible with spherical symmetry. They are naturally classified as *electric* (F_{eI}) and *magnetic* (F_{mI}) forms, and each of these forms is associated with a certain subset $I = \{i_1, \dots, i_k\}$ ($i_1 < \dots < i_k$) of the set of numbers labelling the factor spaces: $\{i\} = I_0 = \{0, \dots, n\}$. By definition, an electric form F_{eI} carries the coordinate indices u and those of the subspaces \mathbb{M}_i , $i \in I$, whereas a magnetic form F_{mI} is built as a form dual to a possible electric one associated with I . Thus nonzero components of F_{mI} carry coordinate indices of the subspaces \mathbb{M}_i , $i \in \bar{I} := I_0 \setminus I$. One can write:

$$n_{eI} = \text{rank } F_{eI} = d(I) + 1, \quad n_{mI} = \text{rank } F_{mI} = D - n_{eI} = d(\bar{I}) \quad (22)$$

where $d(I) = \sum_{i \in I} d_i = \dim \mathbb{M}_I$, $\mathbb{M}_I := \mathbb{M}_{i_1} \times \dots \times \mathbb{M}_{i_k}$. The index s jointly describes the two types of forms.

If the time axis \mathbb{R}_t belongs to \mathbb{M}_I , we are dealing with a true electric or magnetic form, directly generalizing the Maxwell field in \mathbb{M}_{ext} ; otherwise the F -form behaves in \mathbb{M}_{ext} as an effective scalar or pseudoscalar. Such F -forms will be called *quasiscalar*.

The forms F_s are associated with p -branes as extended sources of the spherically symmetric field distributions, where the brane dimension is $p = d(I_s) - 1$, and $d(I_s)$ is the brane world volume dimension. A natural assumption is that the branes only “live” in extra dimensions, i.e., $0 \notin I_s$, $\forall s$.

The classification of F -forms can be illustrated using as an example $D = 11$ supergravity, representing the low-energy limit of M-theory [28]. The action (21) for the bosonic sector of this theory (truncated by omitting the Chern-Simons term) does not contain scalar fields, and the only

F -form is of rank 4, whose various nontrivial components F_s (elementary F -forms, called simply F -forms according to the above convention) are associated with electric 2-branes [for which $d(I_s) = 3$] and magnetic 5-branes [such that $d(I_s) = 6$, see (22)].

Let us put $d_0 = 2$ and ascribe to the external space-time coordinates the indices $M = t, u, \theta, \phi$ (θ and ϕ are the spherical angles), and let the numbers $i = 2, \dots, 8$ refer to the extra dimensions, each associated with an extra factor space \mathbb{M}_i with the same number (\mathbb{M}_i are thus assumed to be one-dimensional). The number $i = 1$ refers to the time axis, $\mathbb{M}_1 = \mathbb{R}_t$, as stated previously. Here are examples of different kinds of forms:

F_{ut23} is a true electric form, $I = \{123\}$; $\bar{I} = \{045678\}$.

$F_{\theta\phi23}$ is a true magnetic form, $I = \{145678\}$; $\bar{I} = \{023\}$.

F_{u234} is an electric quasiscalar form, $I = \{234\}$; $\bar{I} = \{015678\}$.

$F_{\theta\phi t2}$ is a magnetic quasiscalar form, $I = \{345678\}$; $\bar{I} = \{012\}$.

Under the above assumptions, it is helpful to describe the system in the so-called σ model representation [25]). Namely (see more general and detailed descriptions in [15, 25, 26]), let us choose the harmonic u coordinate in \mathbb{M} ($\nabla^M \nabla_M u = 0$), such that

$$\alpha^0(u) = \sum_{i=0}^n d_i \beta^i \equiv d_0 \beta^0 + \sigma_1(u). \quad (23)$$

We use the notations

$$\sigma_i = \sum_{j=i}^n d_j \beta^j(u), \quad \sigma(I) = \sum_{i \in I} d_i \beta^i(u). \quad (24)$$

Then the combination $\binom{u}{u} + \binom{\theta}{\theta}$ of the Einstein equations, where θ is one of the angular coordinates on \mathbb{S}^{d_0} , has a Liouville form, $\ddot{\alpha} - \ddot{\beta}^0 = (d_0 - 1)^2 e^{2\alpha - 2\beta^0}$ (an overdot means d/du), and is integrated giving

$$e^{\beta^0 - \alpha^0} = (d_0 - 1)s(k, u), \quad s(k, u) := \begin{cases} k^{-1} \sinh ku, & k > 0, \\ u, & k = 0, \\ k^{-1} \sin ku, & k < 0. \end{cases} \quad (25)$$

where k is an integration constant. Another integration constant is suppressed by properly choosing the origin of u . With (25) the D -dimensional line element may be written in the form

$$ds_D^2 = \frac{e^{-2\sigma_1/\bar{d}}}{[\bar{d}s(k, u)]^{2/\bar{d}}} \left\{ \frac{du^2}{[\bar{d}s(k, u)]^2} + d\Omega_{d_0}^2 \right\} - e^{2\beta^1} dt^2 + \sum_{i=2}^n e^{2\beta^i} ds_i^2, \quad (26)$$

$\bar{d} := d_0 - 1$. The range of the u coordinate is $0 < u < u_{\max}$ where $u = 0$ corresponds to spatial infinity while u_{\max} may be finite or infinite depending on the form of a particular solution.

The Maxwell-like equations for F_s are integrated in a general form, giving the respective charges $Q_s = \text{const}$. The remaining set of unknowns $\beta^i(u)$, $\varphi^a(u)$ ($i = 1, \dots, n$, $a \in \mathcal{A}$) can be treated as a real-valued vector function $x^A(u)$ (so that $\{A\} = \{1, \dots, n\} \cup \mathcal{A}$) in an $(n + |\mathcal{A}|)$ -dimensional vector space \mathbb{V} (target space). The field equations for x^A can be derived from the Toda-like Lagrangian

$$L = G_{AB} \dot{x}^A \dot{x}^B - V_Q(y), \quad V_Q(y) = - \sum_s \epsilon_s Q_s^2 e^{2y_s} \quad (27)$$

(where $\epsilon_s = 1$ for true electric and magnetic form and $\epsilon_s = -1$ for quasiscalar forms), with the “energy” constraint

$$E = G_{AB} \dot{x}^A \dot{x}^B + V_Q(y) = \frac{d_0}{d_0 - 1} k^2 \text{sign } k, \quad (28)$$

The nondegenerate symmetric matrix

$$(G_{AB}) = \begin{pmatrix} d_i d_j / \bar{d} + d_i \delta_{ij} & 0 \\ 0 & \delta_{ab} \end{pmatrix} \quad (29)$$

specifies a positive-definite metric in \mathbb{V} ; the functions $y_s(u)$ are defined as scalar products:

$$y_s = \sigma(I_s) - \chi_s \bar{\lambda}_s \bar{\varphi} \equiv Y_{s,A} x^A, \quad (Y_{s,A}) = \left(d_i \delta_{iI_s}, \quad -\chi_s \lambda_{sa} \right), \quad (30)$$

where $\delta_{iI} = 1$ if $i \in I$ and $\delta_{iI} = 0$ otherwise; χ_s distinguish electric and magnetic forms: $\chi_{eI} = 1$, $\chi_{mI} = -1$. The contravariant components and scalar products of \vec{Y}_s are found using the matrix G^{AB} inverse to G_{AB} :

$$(G^{AB}) = \begin{pmatrix} \delta^{ij}/d_i - 1/(D-2) & 0 \\ 0 & \delta^{ab} \end{pmatrix}, \quad (Y_s^A) = \left(\delta_{iI} - \frac{d(I)}{D-2}, \quad -\chi_s \lambda_{sa} \right); \quad (31)$$

$$Y_{s,A} Y_{s'}^A \equiv \vec{Y}_s \vec{Y}_{s'} = d(I_s \cap I_{s'}) - \frac{d(I_s)d(I_{s'})}{D-2} + \chi_s \chi_{s'} \bar{\lambda}_s \bar{\lambda}_{s'}. \quad (32)$$

The equations of motion in terms of \vec{Y}_s read

$$\ddot{x}^A = \sum_s q_s Y_s^A e^{2y_s}, \quad q_s := \epsilon_s Q_s^2. \quad (33)$$

One can notice that the metric (29) is quite similar to (19), the metric of \mathbb{T}_φ , especially if $h_{ab} = \delta_{ab}$. The difference is that in G_{AB} given by (29) we have $i, j = \overline{1, n}$ since \mathbb{V} includes as a coordinate the metric function β^1 , whereas in H_{KL} we have $i, j = \overline{2, n}$, so that \mathbb{T}_φ is a subspace of \mathbb{V} .

4.2. Some exact solutions. Black holes

The integrability of the Toda-like system (27) depends on the set of vectors \vec{Y}_s . Each \vec{Y}_s consists of input parameters of the problem and represents an F -form F_s with a nonzero charge Q_s , i.e., one of charged p -branes.

The simplest case of integrability takes place when \vec{Y}_s are mutually orthogonal in \mathbb{V} [26], that is,

$$\vec{Y}_s \vec{Y}_{s'} = \delta_{ss'} Y_s^2, \quad Y_s^2 = d(I)[1 - d(I)/(D-2)] + \bar{\lambda}_s^2 > 0 \quad (34)$$

where $\bar{\lambda}_s^2 = \sum_a \lambda_{sa}^2$. Then the functions $y_s(u)$ obey the decoupled Liouville equations $\ddot{y}_s = \epsilon_s Q_s^2 Y_s^2 e^{2y_s}$, whence

$$e^{-2y_s(u)} = \begin{cases} Q_s^2 Y_s^2 s^2(h_s, u + u_s), & \epsilon_s = 1, \\ Q_s^2 Y_s^2 h_s^{-2} \cosh^2[h_s(u + u_s)], & \epsilon_s = -1, \quad h_s > 0, \end{cases} \quad (35)$$

where h_s and u_s are integration constants and the function $s(.,.)$ has been defined in (25). For the sought-for functions $x^A(u)$ and the “conserved energy” E we then obtain:

$$x^A(u) = \sum_s \frac{Y_s^A}{Y_s^2} y_s(u) + c^A u + \underline{c}^A, \quad (36)$$

$$E = \sum_s \frac{h_s^2 \operatorname{sign} h_s}{Y_s^2} + \bar{c}^2 = \frac{d_0}{d_0 - 1} k^2 \operatorname{sign} k, \quad (37)$$

where the vectors of integration constants \vec{c} and \vec{c} are orthogonal to each \vec{Y}_s : $c^A Y_{s,A} = \underline{c}^A Y_{s,A} = 0$.

Although many other solutions are known [24, 25, 51], their physical properties turn out to be quite similar to those of the present solutions for orthogonal systems (OS) of vectors \vec{Y}_s [51, 52].

Black holes (BHs) are distinguished among other spherically symmetric solutions by the requirement that there should be horizons rather than singularities in \mathbb{M}_{ext} at $u = u_{\text{max}}$. This leads to constraints upon the input and integration constants. The above OS solutions describe BHs if

$$h_s = k > 0, \quad \forall s; \quad c^A = k \sum_s Y_s^{-2} Y_s^A - k \delta_1^A, \quad (38)$$

where $A = 1$ corresponds to $i = 1$ (time). The constraint (37) then holds automatically. The value $u = u_{\text{max}} = \infty$ corresponds to the horizon. The no-hair theorem of Ref. [52] states that BHs are incompatible with quasiscalar F -forms, so that all $\epsilon_s = 1$.

Under the asymptotic conditions $\varphi^a \rightarrow 0$, $\beta^i \rightarrow 0$ as $u \rightarrow 0$, after the transformation

$$e^{-2ku} = 1 - \frac{2k}{\bar{d}r^{\bar{d}}}, \quad \bar{d} := d_0 - 1 \quad (39)$$

the metric (26) for BHs and the corresponding scalar fields may be written as

$$ds_D^2 = \left(\prod_s H_s^{A_s} \right) \left[-dt^2 \left(1 - \frac{2k}{\bar{d}r^{\bar{d}}} \right) \prod_s H_s^{-2/Y_s^2} + \left(\frac{dr^2}{1 - 2k/(\bar{d}r^{\bar{d}})} + r^2 d\Omega^2 \right) + \sum_{i=2}^n ds_i^2 \prod_s H_s^{A_s^i} \right];$$

$$A_s := \frac{2}{Y_s^2} \frac{d(I_s)}{D-2}; \quad A_s^i := -\frac{2}{Y_s^2} \delta_{iI_s}; \quad (40)$$

$$\varphi^a = - \sum_s \frac{\lambda_{sa}}{Y_s^2} \ln H_s, \quad (41)$$

where H_s are harmonic functions in $\mathbb{R}_+ \times \mathbb{S}^{d_0}$:

$$H_s(r) = 1 + P_s/(\bar{d}r^{\bar{d}}), \quad P_s := \sqrt{k^2 + Q_s^2 Y_s^2} - k. \quad (42)$$

The subfamily (38), (40)–(42) exhausts all OS BH solutions with $k > 0$ (non-extremal BHs). Extremal BHs, corresponding to minimum mass for given charges (the so-called BPS limit), are obtained either in the limit $k \rightarrow 0$, or directly from (35)–(37) under the conditions $h_s = k = c^A = 0$. The only independent integration constants in the BH solutions are k , related to the observed mass (see below), and the brane charges Q_s .

Other families of solutions, mentioned previously, also contain BH subfamilies. The most general BH solutions are considered in Ref. [53].

4.3. Post-Newtonian parameters and other observables

One cannot exclude that real astrophysical objects (stars, galaxies, quasars, black holes) are essentially multidimensional objects, whose structure is affected by charged p -branes. (It is then unnecessary to assume that the antisymmetric form fields are directly observable, though one of them may manifest itself as the electromagnetic field.)

The **post-Newtonian** (PN) (weak gravity, slow motion) approximation of multidimensional solutions then determines the predictions of the classical gravitational effects: gravitational redshift, light deflection, perihelion advance and time delay (see [4, 10]).

For spherically symmetric configurations, a standard form of the PN metric uses the Eddington parameters β and γ in isotropic coordinates, in which the spatial part is conformally flat [4]:

$$ds_{\text{PN}}^2 = -(1 - 2V + 2\beta V^2)dt^2 + (1 + 2\gamma V)(d\rho^2 + \rho^2 d\Omega^2) \quad (43)$$

where $d\Omega^2$ is the metric on \mathbb{S}^2 , $V = GM/\rho$ the Newtonian potential, G the Newtonian gravitational constant and M the active gravitating mass.

Observations in the Solar system lead to tight constraints on the Eddington parameters [10]:

$$\gamma = 0.99984 \pm 0.0003, \quad \beta = 0.9998 \pm 0.0006. \quad (44)$$

The first restriction results from over VLBI observations [54], the second one from the γ data and an analysis of lunar laser ranging data [55, 56].

For a theory under consideration, the metric (43) should be identified with the asymptotics of the 4D metric in the observational CF. Preserving its choice yet undetermined, we can write according to (26) with $d_0 = 2$:

$$ds_4^* = e^{2f(u)} \left\{ -e^{2\beta_1} dt^2 + \frac{e^{-2\sigma_1}}{s^2(k, u)} \left[\frac{du^2}{s^2(k, u)} + d\Omega^2 \right] \right\} \quad (45)$$

where $f(u)$ is an arbitrary function of u , normalized for convenience to $f(0) = 0$ (not to be confused with $f(\phi)$ that appeared in Sec. 2 and 3). Recall that by our notations $\sigma_1 = \beta^1 + \sigma_2$, the function $s(k, u)$ is defined in Eq. (25), and spatial infinity takes place at $u = 0$.

Passing to isotropic coordinates in (45), one finds that $u = 1/\rho$ up to cubic terms in $1/\rho$, and the decomposition in $1/\rho$ up to $O(\rho^{-2})$, needed for comparison with (43), precisely coincides with the u -decomposition near $u = 0$.

Using this circumstance, it is easy to obtain for the mass and the Eddington parameters corresponding to (45):

$$GM = -\beta^{1'} - f'; \quad \beta = 1 + \frac{1}{2} \frac{\beta^{1''} + f''}{(GM)^2}, \quad \gamma = 1 + \frac{2f' - \sigma_2'}{GM}, \quad (46)$$

where $f' = df/du|_{u=0}$ and similarly for other functions. The expressions (46) are quite general and are applicable to asymptotically flat, static, spherically symmetric solution of any theory where the energy-momentum tensor has the property $T_u^u + T_\theta^\theta = 0$, which leads to the metric (26) and, in particular, to the above solutions of the theory (21).

Two special choices of $f(u)$ can be distinguished. First, if, for some reasons, the 4D Einstein frame is chosen as the observational one, then, according to Eq. (17) with $d_0 = 2$, we have

$$f = f^E = \sigma_2/2. \quad (47)$$

Second, let us try to add the matter action in (21) simply as $\text{const} \cdot \int L_m \sqrt{D} g d^D x$, i.e., like the fermionic terms in the effective action in the field limit of string theory ([27], Eq. (13.1.49)). In the observational frame with the metric $g_{\mu\nu}^*$ the matter action should read simply $\int d^4 x \sqrt{g^*} L_m$. Identifying them, we obtain [20, 21] $g_{\mu\nu}^* = e^{\sigma_2/2} g_{\mu\nu}$, whence

$$f = f^* = \sigma_2/4. \quad (48)$$

The parameter β can be calculated using (46) *directly from the equations of motion (33), without solving them*. This is true for any function f of the form $f = \vec{F}\vec{x}$ where $\vec{F} \in \mathbb{V}$ is a constant vector (i.e., f is a linear combination of β^i and the scalar fields φ^a):

$$\beta - 1 = \frac{1}{2(GM)^2} \sum_s \epsilon_s Q_s^2 (Y_s^1 + \vec{F}\vec{Y}_s) e^{2y_s(0)}. \quad (49)$$

Explicit expressions for M and γ require knowledge of the solutions' asymptotic form. However, there is an exception: due to (47), in the 4D Einstein frame $\gamma = 1$, precisely as in GR, for all p -brane solutions in the general model (21).

Let us also present the quantities $\beta^{1'}$ and σ'_2 , needed for finding GM and γ , for OS BH solutions:

$$\beta^{1'} = -k - \sum_s P_s \frac{1 - b_s}{Y_s^2}, \quad \sigma'_2 = - \sum_s \frac{1 - 2b_s}{Y_s^2} \quad (50)$$

where $b_s = d(I_s)/(D - 2)$.

Some general observations can be made from the above relations [21]:

- The expressions for β depend on the input constants D , $d(I_s)$ (hence on p -brane dimensions), on the mass M and on the charges Q_s . For given M , they are independent of other integration constants, emerging in the solution of the Toda system (33), and also of p -brane intersection dimensions, since they are obtained directly from Eqs.(33) [21]. This means, in particular, that β is the same for BH and non-BH configurations with the same set of input parameters, mass and charges.
- All p -branes give positive contributions to β in both frames (47) and (48), which leads to a general restriction on the charges Q_s for given mass and input parameters.
- The expressions for γ depend, in general, on the integration constants h_s and c^i emerging from solving Eqs.(33). For BH solutions these constants are expressed in terms of k and the input parameters, so both β and γ depend on the mass, charges and input parameters.
- In the 4-E frame, one always has $\gamma = 1$. The same is true for some BH solutions in all frames with $f = N\sigma_2$, $N = \text{const}$ [21].

BH temperature. BHs are, like nothing else, strong-field gravitational objects, while the PN parameters only describe their far neighbourhood. An important observable characteristic of their strong-field behaviour is the Hawking temperature T_H . One can show [21] that this quantity is *CF-independent*, at least if conformal factors that connect different frames are regular on the horizon. The conformal invariance of T_H was also discussed in another context in Ref. [57].

In particular, for the above OS BH solutions one obtains [26]

$$T_H = \frac{1}{8\pi k k_B} \prod_s \left(\frac{2k}{2k + P_s} \right)^{1/Y_s^2}. \quad (51)$$

where k_B is the Boltzmann constant.

The physical meaning of T_H is related to quantum evaporation, a process to be considered in the fundamental frame, while the produced particles are assumed to be observed at flat infinity, where relevant CFs do not differ. Therefore T_H should be CF-independent, and this property is obtained “by construction” [21].

All this is true for T_H in terms of the integration constant k and the charges Q_s . However, the observed mass M as a function of the same quantities is frame-dependent, see (46). Therefore T_H as a function of M and Q_s is frame-dependent as well.

Coulomb law violation is one of specific potentially observable effects of extra dimensions. Suppose in (15), (14) $d_0 = 2$ and let us try to describe the electrostatic field of a spherically symmetric source by a term $F^2 e^{2\bar{\lambda}\varphi}$ in the action (21), corresponding to a true electric m -form F_{eI} with a certain set I containing 1, that is, $I = 1 \cup J$, $J \subset \{2, \dots, n\}$.

Then the modified Coulomb law in any CF with the metric (45) can be written as follows [21]:

$$E == (|Q|/r^2) e^{-2\bar{\lambda}\bar{\varphi} + \sigma(J) - \sigma(\bar{J})}, \quad (52)$$

where E is the observable electric field strength, r is the observable radius of coordinate spheres, the notations (24) are used and $\bar{J} = \{2, \dots, n\} \setminus J$.

Deviations from the conventional Coulomb law are evidently both due to extra dimensions (and depend on the F -form structure) and due to interaction of F_s with the scalar fields. This relation (generalizing the one obtained in Ref. [14] in the framework of dilaton gravity) is valid for an arbitrary metric of the form (14) ($d_0 = 2$) and does not depend on whether or not this F -form takes part in the formation of the gravitational field.

Eq. (52) is exact and — which is remarkable — it is *CF-independent*. This is an evident manifestation of the conformal invariance of the electromagnetic field in \mathbb{M}_{ext} even in the present generalized framework.

5. Concluding remarks

We have seen that the properties of theoretical models look drastically different when taken in different CFs. This once again stresses the necessity of a careful reasoning for a particular choice of a CF. It even may happen (though seems unlikely) that there is an unobservable part of the Universe, separated from us by a singularity in the observational CF which is converted to a regular surface (S_{trans}) after passing to a fundamental frame. A similar thing may happen to a cosmological singularity: in some theories it can correspond to a regular bounce in a fundamental frame.

We have obtained expressions for the Eddington PN parameters β and γ for a wide range of static, spherically symmetric solutions of multidimensional gravity with the general string-inspired action (21). The experimental limits (44) on β and γ constrain certain combinations of the solution parameters. This, however, concerns only the particular physical system for which the measurements are carried out, in our case, the Sun's gravitational field. The main feature of the expressions for β and γ is their dependence not only on the theory (the constants entering into the action), but on the particular solution (integration constants). The PN parameters thus can be different for different self-gravitating systems, and not only, say, for stars and black holes, but even for different stars if we try to describe their external fields in models like (21).

A feature of interest is the universal prediction of $\beta > 1$ in (49) for both frames (47) and (48). The predicted deviations of γ from unity may be of any sign and depend on many integration constants. If, however, the 4-dimensional Einstein frame is adopted as the observational one, we have a universal result $\gamma = 1$ for all static, spherically symmetric solutions of the theory (21).

The BH temperature T_{H} also carries information about the space-time structure, encoded in Y_s^2 . Being a universal parameter of a given solution to the field equations, T_{H} as a function of the observable BH mass and charges is still CF-dependent due to different expressions for the mass M in different frames.

One more evident consequence of extra dimensions is the Coulomb law violation, caused by a modification of the conventional Gauss theorem and also by scalar-electromagnetic interaction. A remarkable property of the modified Coulomb law is its CF independence for a given static, spherically symmetric metric.

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